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Nondifferentiability of an objective function at its point of minimum for most minimax problems

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Abstract In this paper we study a class of minimax problems $\max\{f(x), g(x)\} \to \min$, $x \in R^n$ where $f, g \in C^1(R^n)$. We show that the subclass of all problems for which there exists a point of minimum $z \in R^1$ such that $f(z) = g(z)$ and $\nabla f(z) = \nabla g(z)$ is small.

Keywords Complete metric space · Minimax problem · Open everywhere dense set

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1 Introduction

The study of minimax problems is one of central topics in optimization theory. See, for example, [2–4] and the references mentioned therein. In this paper we consider a class of minimax problems

$$\max\{f(x), g(x)\} \to \min, \quad x \in R^n$$

described below where the functions $f, g : R^n \to R^1$ are continuously differentiable. We show that the subclass of all problems for which there exists a point of minimum $z \in R^n$ such that

$$f(z) = g(z) \text{ and } \nabla f(z) = \nabla g(z)$$

is small. It means that for a typical minimax problem, if its solution z satisfies $f(z) = g(z)$, then the cost function $\max\{f, g\}$ is not differentiable at z. Thus, in general, methods for minimization of smooth functions cannot be applied for solving minimax problems. Here, instead of considering a certain property for a single minimax problem, we investigate it for

The paper is dedicated to the memory of Alexander M. Rubinov.

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a class of minimax problems and show that this property holds for most of the problems in the class. This approach has already been successfully applied in many areas of Analysis (see, for example, [1,5,8]).

The main result of the paper is a solution of a problem suggested to the author by A. M. Rubinov in 2003. It should be mentioned that partial solutions of this problem were obtained in [6,7,9]. In [6,9] the analogs of our main result were obtained for minimax problems on a real line with $n = 1$. In [7] an analogous result was obtained for pairs of continuously differentiable functions $f, g : R^n \to R^1$ where f is convex.

We denote by $|\cdot|$ the Euclidean norm of the space R^n, by $(\cdot, \cdot)$ the inner product of R^n, for each differentiable function $f : R^n \to R^1$ and each $x \in R^n$ set

$$\nabla f(x) = ((\partial f/\partial x_1)(x), \dots, (\partial f/\partial x_i)(x), \dots, (\partial f/\partial x_n)(x))$$

and denote by $C^1(R^n)$ the set of all continuously differentiable functions $f : R^n \to R^1$.

For each $x \in R^n$ and each $r > 0$ set

$$B(x, r) = \{z \in R^n : |z - x| \le r\}, \ B(r) = B(0, r).$$

Denote by e be an element of R^n all of whose coordinates are unity.

Let X be a nonempty set. For each function $f : X \to R^1$ set $\inf(f) = \inf\{f(x) : x \in X\}$. For each pair of functions $f, g : X \to R^1$ define a function $\max\{f, g\} : X \to R^1$ by

$$\max\{f, g\}(x) = \max\{f(x), g(x)\}, \ x \in X. \tag{1.1}$$

We use the convention that $\infty/\infty = 1$.

For each differentiable function $f : R^n \to R^1$ set

$$||f||_1 = \sup\{|f(z)|, \ |\nabla f(z)| : z \in R^n\}. \tag{1.2}$$

Denote by $\mathcal{M}$ the set of all pairs (f, g) such that $f, g \in C^1(R^n)$ and satisfy the following condition:

$$\lim_{|x| \to \infty} \max\{f(x), g(x)\} = \infty. \tag{1.3}$$

For each pair $(f_1, g_1), (f_2, g_2) \in \mathcal{M}$ define

$$\tilde{d}((f_1, g_1), (f_2, g_2)) = \sup\{||f_1 - f_2||_1, \ ||g_1 - g_2||_1\},$$
$$d((f_1, g_1), (f_2, g_2)) = \tilde{d}((f_1, g_1), (f_2, g_2))(1 + \tilde{d}((f_1, g_1), (f_2, g_2)))^{-1}. \tag{1.4}$$

It is clear that $(\mathcal{M}, d)$ is a complete metric space.

In view of (1.3) for each $(f, g) \in \mathcal{M}$ the minimization problem

$$\max\{f(x), g(x)\} \to \min, \ x \in R^n$$

has a solution.

Denote by $\mathcal{M}_0$ the set of all pairs $(f, g) \in \mathcal{M}$ for which there is $x \in R^n$ such that

$$f(x) = g(x) = \inf(\max\{f, g\}) \tag{1.5}$$

and denote by G the set of all pairs $(f, g) \in \mathcal{M}$ for which there is $x \in R^n$ such that (1.5) holds and

$$\nabla f(x) = \nabla g(x). \tag{1.6}$$

For the proof of the following result see [6, Proposition 1.1].

Proposition 1.1 *$\mathcal{M}_0$ and G are closed subsets of the metric space $(\mathcal{M}, d)$.*

Note that in [6] this result was proved for $n = 1$ but the same proof remains in force for any natural number n.

In this paper we will establish the following result.

Theorem 1.1 *$\mathcal{M}_0 \backslash G$ is an open everywhere dense subset of $(\mathcal{M}_0, d)$ and $\mathcal{M} \backslash G$ is an open everywhere dense subset of $(\mathcal{M}, d)$.*

2 Proof of Theorem 1.1

In the sequel we use the following auxiliary result [2].

Lemma 2.1 *Let $(f, g) \in G$, $x \in R^n$, and let*

$$f(x) = g(x) = \inf(\max\{f, g\}),\ \nabla f(x) = \nabla g(x).$$

Then $\nabla f(x) = \nabla g(x) = 0$.

By Proposition 1.1 the set $\mathcal{M} \backslash G$ is an open subset of $(\mathcal{M}, d)$ and $\mathcal{M}_0 \backslash G$ is an open subset of $(\mathcal{M}_0, d)$. In order to prove Theorem 1.1 it is sufficient to show that $\mathcal{M} \backslash G$ is an everywhere dense subset of $(\mathcal{M}, d)$ and $\mathcal{M}_0 \backslash G$ is an everywhere dense subset of $(\mathcal{M}_0, d)$. Theorem 1.1 follows easily from the following result.

Proposition 2.1 *Let $(f, g) \in G$, $\bar{x} \in R^n$,*

$$f(\bar{x}) = g(\bar{x}) = \inf(\max\{f, g\}), \tag{2.1}$$

$$\nabla f(\bar{x}) = \nabla g(\bar{x}) \tag{2.2}$$

and let $\mathcal{U}$ be an open neighborhood of (f, g) in $(\mathcal{M}, d)$. Then there exists $(\tilde{f}, \tilde{g}) \in (\mathcal{U} \cap \mathcal{M}_0) \backslash G$.

Proof By (2.1), (2.2) and Lemma 2.1,

$$\nabla f(\bar{x}) = \nabla g(\bar{x}) = 0. \tag{2.3}$$

We construct the pair $(\tilde{f}, \tilde{g})$ with the three steps. In the first step we perturb the functions f, g and construct a pair $(f_0, g_0) \in \mathcal{U}$ such that

$$f_0(\bar{x}) = f(\bar{x}) = g(\bar{x}) = g_0(\bar{x}),$$
$$\nabla f_0(\bar{x}) = \nabla g_0(\bar{x}) = 0$$

and that $\bar{x}$ is a unique solution of the minimization problem

$$(\max\{f_0, g_0\})(x) \to \min, \quad x \in R^n.$$

In the second step we perturb the functions f_0, g_0 and construct a pair $(f_1, g_1) \in \mathcal{U}$ such that

$$f_1(\bar{x}) = g_1(\bar{x}),\ \nabla f_1(\bar{x}) = \nabla g_1(\bar{x}) = 0,$$

$\bar{x}$ is a unique solution of the minimization problem

$$(\max\{f_1, g_1\})(x) \to \min, \quad x \in R^n$$

and that for all x belonging to a small neighborhood of $\bar{x}$

$$f_1(x) = f_1(\bar{x}) + |x - \bar{x}|^2, g_1(x) = g_1(\bar{x}) + |x - \bar{x}|^2.$$

In the third step we perturb the functions f_1, g_1 and construct a pair $(\tilde{f}, \tilde{g}) \in (\mathcal{U} \cap \mathcal{M}_0)\setminus G$ in such a way that

$$\tilde{f}(\bar{x}) = \tilde{g}(\bar{x}),$$

$\bar{x}$ is a unique solution of the minimization problem

$$(\max\{\tilde{f}, \tilde{g}\})(x) \to \min, \quad x \in R^n$$

and that for all x belonging to a small neighborhood of $\bar{x}$

$$\tilde{f}(x) = \tilde{f}(\bar{x}) + |x - \bar{x}|^2 + \kappa(e, x - \bar{x}), \ \tilde{g}(x) = \tilde{g}(\bar{x}) + |x - \bar{x}|^2 + \kappa(e, \bar{x} - x),$$

where κ is a small positive number and $e = (1, 1, \dots, 1)$.

There is a function $\phi : R^1 \to [0, 1]$ such that $\phi \in C^\infty(R^1)$,

$$\phi(t) = 1 \quad \text{if } |t| \leq 2^{-1}, \ \phi(t) = 0 \quad \text{if } |t \geq 1, \\ 0 < \phi(t) < 1 \quad \text{if } 2^{-1} < |t| < 1. \tag{2.4}$$

By (1.3) there is $d_0 > 1$ such that

$$\max\{f(x), g(x)\} > f(\bar{x}) + 8 \quad \text{for each } x \in R^n \setminus B(\bar{x}, d_0/2). \tag{2.5}$$

Choose a positive number

$$c_1 \leq (2d_0^2)^{-1} \tag{2.6}$$

and set

$$\psi_0(x) = |x - \bar{x}|^2 \phi(c_1 |x - \bar{x}|^2), x \in R^n. \tag{2.7}$$

It is clear that for all $x \in R^1$

$$\psi_0(x) = 0 \quad \text{if } |x - \bar{x}| \geq c_1^{-1/2}, \tag{2.8}$$

$$\psi_0(x) = |x - \bar{x}|^2 \quad \text{if } |x - \bar{x}| \leq d_0. \tag{2.9}$$

It is not difficult to see that there exists a positive number c_0 such that the pair of function (f_0, g_0) defined by

$$f_0(x) = f(x) + c_0\psi_0(x), \ g_0(x) = g(x) + c_0\psi_0(x), \quad x \in R^n \tag{2.10}$$

belongs to $\mathcal{U}$. By (2.10), (2.9) and (2.1),

$$f_0(\bar{x}) = f(\bar{x}) = g(\bar{x}) = g_0(\bar{x}). \tag{2.11}$$

In view of (2.10), (2.3) and (2.9),

$$\nabla f_0(\bar{x}) = \nabla f(\bar{x}) + c_0 \nabla \psi_0(\bar{x}) = 0, \\ \nabla g_0(\bar{x}) = \nabla g(\bar{x}) + c_0 \nabla \psi_0(\bar{x}) = 0. \tag{2.12}$$

Assume that $\{x^{(k)}\}_{k=1}^\infty \subset R^n$ satisfies

$$\lim_{k \to \infty} \max\{f_0(x^{(k)}), g_0(x^{(k)})\} = \inf(\max\{f_0, g_0\}). \tag{2.13}$$

We show that

$$\lim_{k\to\infty} x^{(k)} = \bar{x}. \tag{2.14}$$

Since $(f_0, g_0) \in \mathcal{M}$ it follows from (1.3) that the sequence $\{x^{(k)}\}_{k=1}^{\infty}$ is bounded. We may assume without loss of generality that there exists

$$\widehat{x} = \lim_{k\to\infty} x^{(k)}. \tag{2.15}$$

It is sufficient to show that $\widehat{x} = \bar{x}$. By (2.15) and (2.13),

$$\max\{f_0(\widehat{x}), g_0(\widehat{x})\} = \inf(\max\{f_0, g_0\}). \tag{2.16}$$

It follows from (2.16), (2.10), (2.1), (2.11) and (2.7) that

$$\begin{aligned}\inf(\max\{f_0, g_0\}) &= \max\{f(\widehat{x}), g(\widehat{x})\} + c_0\psi_0(\widehat{x}) \geq \max\{f(\bar{x}), g(\bar{x})\} + c_0\psi_0(\widehat{x})\\ &= \max\{f_0(\bar{x}), g_0(\bar{x})\} + c_0\psi_0(\widehat{x})\end{aligned}$$

and

$$\psi_0(\widehat{x}) = 0. \tag{2.17}$$

In view of (2.10), (2.7), (2.16), (2.11) and (2.1),

$$\begin{aligned}\max\{f(\widehat{x}), g(\widehat{x})\} &\leq \max\{f_0(\widehat{x}), g_0(\widehat{x})\} \leq \max\{f_0(\bar{x}), g_0(\bar{x})\}\\ &= \max\{f(\bar{x}), g(\bar{x})\} = f(\bar{x}).\end{aligned}$$

Together with (2.5) these relations imply that $|\widehat{x} - \bar{x}| \leq d_0/2$.

Combined with (2.17) and (2.9) this inequality implies that $\bar{x} = \widehat{x}$. Therefore we have shown that the following property holds:

(P1) If the sequence $\{x^{(k)}\}_{k=1}^{\infty} \subset R^n$ satisfies (2.13), then $\lim_{k\to\infty} x^{(k)} = \bar{x}$.

For each $\gamma \in (0, 1)$ set

$$\psi_\gamma(x) = \phi(\gamma^{-2}|x - \bar{x}|^2),\ \ x \in R^n. \tag{2.18}$$

It is clear that for each $\gamma \in (0, 1)$, $\psi_\gamma \in C^\infty(R^n)$ and for all $x \in R^n$

$$\psi_\gamma(x) = 0 \text{ if } |x - \bar{x}| \geq \gamma \text{ and } \psi_\gamma(x) = 1 \text{ if } |x - \bar{x}| \leq \gamma(2^{-1/2}). \tag{2.19}$$

Since $(f_0, g_0) \in \mathcal{U}$ there is $r_0 > 0$ such that

$$\{(\xi_1, \xi_2) \in \mathcal{M} :\ \tilde{d}((\xi_1, \xi_2), (f_0, g_0)) \leq r_0\} \subset \mathcal{U}. \tag{2.20}$$

Since $f_0, g_0 \in C^1(R^n)$ it follows from (2.4) and (2.12) that there exists $\gamma_1 \in (0, 1)$ such that

$$\begin{aligned}&[2\gamma_1 + 2\sup\{|\nabla f_0(z)| + |\nabla g_0(z)| + |f_0(\bar{x}) - f_0(z)| + |g_0(\bar{x}) - g_0(z)| :\\ &z \in R^n \text{ and } |z - \bar{x}| \leq \gamma_1\}](1 + \sup\{|\phi'(z)| :\ z \in R^n\}) < (8n)^{-1} r_0.\end{aligned} \tag{2.21}$$

Define

$$\begin{aligned}f_1(x) &= \psi_{\gamma_1}(x)(|x - \bar{x}|^2 + f_0(\bar{x})) + (1 - \psi_{\gamma_1}(x)) f_0(x), \quad x \in R^n,\\ g_1(x) &= \psi_{\gamma_1}(x)(|x - \bar{x}|^2 + g_0(\bar{x})) + (1 - \psi_{\gamma_1}(x)) g_0(x), \quad x \in R^n.\end{aligned} \tag{2.22}$$

Clearly, $f_1, g_1 \in C^1(R^n)$ and

$$\lim_{|x|\to\infty} \max\{f_1(x), g_1(x)\} = \infty.$$

Thus $(f_1, g_1) \in \mathcal{M}$. Relations (2.22), (2.19) and (2.11) imply that

$$f_1(\bar{x}) = f_0(\bar{x}) = g_0(\bar{x}) = g_1(\bar{x}). \tag{2.23}$$

We show that the following property holds:

(P2) If a sequence $\{x^{(k)}\}_{k=1}^{\infty} \subset R^n$ satisfies

$$\lim_{k\to\infty} \max\{f_1(x^{(k)}), g_1(x^{(k)})\} = \inf(\max\{f_1, g_1\}), \tag{2.24}$$

then $\lim_{k\to\infty} x^{(k)} = \bar{x}$.

Assume that a sequence $\{x^{(k)}\}_{k=1}^{\infty} \subset R^n$ satisfies (2.24). By (2.24) and the growth condition (1.3) the sequence $\{x^{(k)}\}_{k=1}^{\infty}$ is bounded. We may assume without loss of generality that there exists

$$\tilde{x} = \lim_{k\to\infty} x^{(k)}. \tag{2.25}$$

It is sufficient to show that $\tilde{x} = \bar{x}$. In view of (2.25) and (2.24)

$$\max\{f_1(\tilde{x}),\ g_1(\tilde{x})\} = \inf(\max\{f_1, g_1\}). \tag{2.26}$$

Assume that $\tilde{x} \neq \bar{x}$. Then it follows from this relation, (P1) and (2.11) that

$$\max\{f_0(\tilde{x}),\ g_0(\tilde{x})\} > f_0(\bar{x}) = g_0(\bar{x}). \tag{2.27}$$

It follows from (2.22), (2.11), (2.18), (2.4), (2.27), (2.23) and the relation $\tilde{x} \neq \bar{x}$ that

$$\begin{aligned}\max\{f_1(\tilde{x}), g_1(\tilde{x})\} &= \psi_{\gamma_1}(\tilde{x})(|\tilde{x} - \bar{x}|^2 + f_0(\bar{x})) + (1 - \psi_{\gamma_1}(\tilde{x}))\max\{f_0(\tilde{x}), g_0(\tilde{x})\}\\ &> f_0(\bar{x}) = g_0(\bar{x}) = f_1(\bar{x}) = g_1(\bar{x}).\end{aligned}$$

This relation contradicts (2.26). The contradiction we have reached proves that $\tilde{x} = \bar{x}$. Thus we have shown that (P2) holds.

By (2.22) and (2.19)

$$\nabla f_1(\bar{x}) = \nabla g_1(\bar{x}) = 0. \tag{2.28}$$

In view of (2.22), (2.19), (2.4), (2.18) and (2.21)

$$\begin{aligned}&\sup\{|f_1(x) - f_0(x)|,\ |g_1(x) - g_0(x)| :\ x \in R^n\}\\ &\quad= \sup\{\psi_{\gamma_1}(x)||x-\bar{x}|^2 + f_0(\bar{x}) - f_0(x)|,\ \psi_{\gamma_1}(x)||x-\bar{x}|^2 + g_0(\bar{x}) - g_0(x)| :\ x \in R^n\}\\ &\quad\le \sup\{|x-\bar{x}|^2 + |f_0(\bar{x}) - f_0(x)|,\ |x-\bar{x}|^2 + |g_0(\bar{x}) - g_0(x)| :\ x \in R^n \text{ and } |x-\bar{x}| \le \gamma_1\}\\ &\quad\le \gamma_1^2 + \sup\{|f_0(\bar{x}) - f_0(x)|,\ |g_0(\bar{x}) - g_0(x)| : x \in R^n \text{ and } |x - \bar{x}| \le \gamma_1\} < (8n)^{-1} r_0.\end{aligned} \tag{2.29}$$

Let $x \in R^n$ and fix $i \in \{1, \dots, n\}$. By (2.22)

$$\begin{aligned}(\partial f_1/\partial x_i)(x) - (\partial f_0/\partial x_i)(x) &= (\partial(f_1 - f_0)/\partial x_i)(x)\\ &= (\partial\psi_{\gamma_1}/\partial x_i)(x)(|x - \bar{x}|^2 + f_0(\bar{x})) + \psi_{\gamma_1}(x)2(x_i - \bar{x}_i)\\ &\quad - (\partial\psi_{\gamma_1}/\partial x_i)(x) f_0(x) - \psi_{\gamma_1}(x)(\partial f_0/\partial x_i)(x)\\ &= (\partial\psi_{\gamma_1}/\partial x_i)(x)(|x - \bar{x}|^2 + f_0(\bar{x}) - f_0(x))\\ &\quad + \psi_{\gamma_1}(x)(2(x_i - \bar{x}_i) - (\partial f_0/\partial x_i)(x)).\end{aligned}$$

Together with (2.19), (2.18) and (2.4) this implies that

$$\begin{aligned}
&|(\partial f_1/\partial x_i)(x) - (\partial f_0/\partial x_i)(x)| \\
&\quad \le |(\partial \psi_{\gamma_1}/\partial x_i)(x)|||x-\bar{x}|^2 + f_0(\bar{x}) - f_0(x)| \\
&\qquad + \psi_{\gamma_1}(x)|2(x_i - \bar{x}_i) - (\partial f_0/\partial x_i)(x)| \\
&\quad \le |(\partial \psi_{\gamma_1}/\partial x_i)(x)|||x-\bar{x}|^2 + f_0(\bar{x}) - f_0(x)| \\
&\qquad + \sup\{|2(z_i - \bar{x}_i) - (\partial f_0/\partial x_i)(z) : z \in R^n \text{ and } |z-\bar{x}| \le \gamma_1\} \\
&\quad \le |(\partial \psi_{\gamma_1}/\partial x_i)(x)|||x-\bar{x}|^2 + f_0(\bar{x}) - f_0(x)| \\
&\qquad + 2\gamma_1 + \sup\{|(\partial f_0/\partial x_i)(z)| : z \in R^n \text{ and } |z-\bar{x}| \le \gamma_1\}.
\end{aligned} \tag{2.30}$$

In view of (2.18)

$$(\partial \psi_{\gamma_1}/\partial x_i)(x) = \phi'(|x-\bar{x}|^2\gamma_1^{-2})2(x_i-\bar{x}_i)\gamma_1^{-2}. \tag{2.31}$$

There are two cases: $|x-\bar{x}| > \gamma_1$; $|x-\bar{x}| \le \gamma_1$.

If $|x-\bar{x}| > \gamma_1$, then it follows from (2.19) and (2.30) that

$$\begin{aligned}
&|(\partial f_1/\partial x_i)(x) - (\partial f_0/\partial x_i)(x)| \\
&\quad \le 2\gamma_1 + \sup\{|(\partial f_0/\partial x_i)(z)| : z \in R^n \text{ and } |z-\bar{x}| \le \gamma_1\}.
\end{aligned} \tag{2.32}$$

Assume that

$$|x-\bar{x}| \le \gamma_1. \tag{2.33}$$

By the mean-value theorem there is $\theta \in [0, 1]$ such that

$$f_0(x) - f_0(\bar{x}) = (\nabla f_0(\bar{x} + \theta(x-\bar{x})), x-\bar{x}). \tag{2.34}$$

Relations (2.33) and (2.34) imply that

$$\begin{aligned}
|f_0(x) - f_0(\bar{x})| &\le |x-\bar{x}||\nabla f_0(\bar{x}+\theta(x-\bar{x}))| \\
&\le \gamma_1 \sup\{|\nabla f_0(z)| : z \in R^n \text{ and } |z-\bar{x}| \le \gamma_1\}.
\end{aligned}$$

It follows from this inequality, (2.31) and (2.33) that

$$\begin{aligned}
&|(\partial \psi_{\gamma_1}/\partial x_i)(x)|||x-\bar{x}|^2 + f_0(\bar{x}) - f_0(x)| \\
&\quad = |\phi'(|x-\bar{x}|^2\gamma_1^{-2})|2\gamma_1^{-1}(\gamma_1^2 + \gamma_1 \sup\{|\nabla f_0(z)| : z \in R^n \text{ and } |z-\bar{x}| \le \gamma_1\}) \\
&\quad = |\phi'(|x-\bar{x}|^2\gamma_1^{-2})|(2\gamma_1 + 2\sup\{|\nabla f_0(z)| : z \in R^n \text{ and } |z-\bar{x}| \le \gamma_1\}) \\
&\quad \le \sup\{|\phi'(z)| : z \in R^1\}(2\gamma_1 + 2\sup\{|\nabla f_0(z)| : z \in R^n \text{ and } |z-\bar{x}| \le \gamma_1\}).
\end{aligned}$$

Combined with (2.30) this implies that

$$\begin{aligned}
&|(\partial f_1/\partial x_i(x) - (\partial f_0/\partial x_i)(x)| \\
&\quad \le \sup\{|\phi'(z)| : z \in R^1\}(2\gamma_1 + 2\sup\{|\nabla f_0(z)| : z \in R^n \text{ and } |z-\bar{x}| \le \gamma_1\}) \\
&\qquad + 2\gamma_1 + \sup\{|\nabla f_0(z)| : z \in R^n \text{ and } |z-\bar{x}| \le \gamma_1\} \\
&\quad \le [2\gamma_1 + 2\sup\{|\nabla f_0(z)| : z \in R^n \text{ and } |z-\bar{x}| \le \gamma_1\}](1 + \sup\{|\phi'(z)| : z \in R^1\}).
\end{aligned}$$

Combined with (2.32) this inequality implies that in both cases

$$\begin{aligned}
&|(\partial f_1/\partial x_i)(x) - (\partial f_0/\partial x_i)(x)| \\
&\quad \le [2\gamma_1 + 2\sup\{|\nabla f_0(z)| : z \in R^n \text{ and } |z-\bar{x}| \le \gamma_1\}](1 + \sup\{|\phi'(z)| : z \in R^1\}).
\end{aligned}$$

Since the relation above holds for all $x \in R^n$ and all $i = 1, \dots, n$ it follows from (2.21) that

$$\begin{aligned}&\sup\{|(\partial f_1/\partial x_i)(x) - (\partial f_0/\partial x_i)(x)| : x \in R^n, \ i = 1, \dots, n\}\\ &\quad \le [2\gamma_1 + 2\sup\{|\nabla f_0(z)| : z \in R^n \text{ and } |z - \bar{x}| \le \gamma_1\}](1 + \sup\{|\phi\prime(z)| : z \in R^1\})\\ &\quad < (8n)^{-1} r_0. \end{aligned} \tag{2.35}$$

Analogously we can show that

$$\begin{aligned}&\sup\{|(\partial g_1/\partial x_i)(x) - (\partial g_0/\partial x_i)(x))| : x \in R^n, \ i = 1, \dots, n\}\\ &\quad \le [2\gamma_1 + 2\sup\{|\nabla g_0(z)| : z \in R^n \text{ and } |z - \bar{x}| \le \gamma_1\}](1\\ &\qquad + \sup\{|\phi'(z)| : z \in R^1\}) < (8n)^{-1} r_0. \end{aligned} \tag{2.36}$$

By (2.35) and (2.36)

$$\tilde{d}((f_1, g_1), (f_0, g_0)) < r_0.$$

Together with (2.20) this implies that

$$(f_1, g_1) \in \mathcal{U}. \tag{2.37}$$

Therefore we have constructed $(f_1, g_1) \in \mathcal{U}$ which satisfies (2.23) and (2.28) and for which property (P2) holds.

It follows from (2.19) and (2.22) that for each $x \in B(\bar{x}, 2^{-1}\gamma_1)$

$$f_1(x) = f_0(\bar{x}) + |x - \bar{x}|^2, \ \ g_1(x) = g_0(\bar{x}) + |x - \bar{x}|^2. \tag{2.38}$$

Recall that $e = (1, 1, 1 \dots 1) \in R^n$ is an element of R^n all of whose coordinates are unity. Choose a positive number γ_2 such

$$(2\gamma_2)^{-1/2} < 2^{-1}\gamma_1. \tag{2.39}$$

It follows from the property (P2) and the growth condition (1.3) which holds for (f_1, g_1) that there is $\Delta > 0$ such that

$$\max\{f_1(x), g_1(x)\} > \max\{f_1(\bar{x}), g_1(\bar{x})\} + \Delta \text{ for each } x \in R^n \backslash B(\bar{x}, (4\gamma_2)^{-1/2}). \tag{2.40}$$

It is not difficult to see that there exists a positive number

$$\gamma_3 < \Delta\gamma_2^{1/2}(4n)^{-1} \tag{2.41}$$

such that the pair of functions (f_2, g_2) defined by

$$\begin{aligned} f_2(x) &= f_1(x) + \gamma_3(e, x - \bar{x})\phi(|x - \bar{x}|^2\gamma_2), \quad x \in R^n,\\ g_2(x) &= g_1(x) - \gamma_3(e, x - \bar{x})\phi(|x - \bar{x}|^2\gamma_2), \quad x \in R^n \end{aligned} \tag{2.42}$$

belongs to $\mathcal{U}$. In view of (2.23) and (2.42)

$$f_2(\bar{x}) = f_1(\bar{x}) = g_1(\bar{x}) = g_2(\bar{x}). \tag{2.43}$$

By (2.42) and (2.4) for each $x \in R^n$ satisfying $|x - \bar{x}| \le (2\gamma_2)^{-1/2}$,

$$f_2(x) = f_1(x) + \gamma_3(e, x - \bar{x}), \ \ g_2(x) = g_1(x) + \gamma_3(e, \bar{x} - x). \tag{2.44}$$

It follows from (2.42) and (2.4) that for each $x \in R^n$ satisfying $|x - \bar{x}| \ge \gamma_2^{-1/2}$,

$$f_2(x) = f_1(x), \ \ g_2(x) = g_1(x). \tag{2.45}$$

Relations (2.44) and (2.28) imply that

$$\nabla f_2(\bar{x}) = \nabla f_1(\bar{x}) + \gamma_3 e,$$

$$\nabla g_2(\bar{x}) = \nabla g_1(\bar{x}) - \gamma_3 e = \nabla f_1(\bar{x}) - \gamma_3 e = \nabla f_2(\bar{x}) - 2\gamma_3 e. \tag{2.46}$$

Let $x \in R^n \setminus \{\bar{x}\}$. We will show that

$$\max\{f_2(x), g_2(x)\} > \max\{f_2(\bar{x}), g_2(\bar{x})\}. \tag{2.47}$$

There are three cases:

$$|x - \bar{x}| \geq \gamma_2^{-1/2}, \quad |x - \bar{x}| \in [(2\gamma_2)^{-1/2},\ \gamma_2^{-1/2}), \quad 0 < |x - \bar{x}| < (2\gamma_2)^{-1/2}.$$

If $|x - \bar{x}| \geq \gamma_2^{-1/2}$, then by (2.45), (2.43) and (P2)

$$\max\{f_2(x), g_2(x)\} = \max\{f_1(x), g_1(x)\} > \max\{f_1(\bar{x}), g_1(\bar{x})\} = \max\{f_2(\bar{x}), g_2(\bar{x})\}$$

and (2.47) is true.

Assume that

$$|x - \bar{x}| \in [(2\gamma_2)^{-1/2}, \gamma_2^{-1/2}]. \tag{2.48}$$

It follows from (2.42), (2.4), (2.48), (2.40), (2.43) and (2.41) that

$$\begin{aligned}
\max\{f_2(x), g_2(x)\} &\geq \max\{f_1(x), g_1(x)\} - \gamma_3|(e, x - \bar{x})| \\
&\geq \max\{f_1(\bar{x}), g_1(\bar{x})\} + \Delta - \gamma_3|(e, x - \bar{x})| \\
&\geq \max\{f_2(\bar{x}), g_2(\bar{x})\} + \Delta - \gamma_3|x - \bar{x}|n \\
&\geq \max\{f_2(\bar{x}), g_2(\bar{x})\} + \Delta - \gamma_3 n \gamma_2^{-1/2} \\
&\geq \max\{f_2(\bar{x}), g_2(\bar{x})\} + \Delta/2
\end{aligned}$$

and (2.47) is true.

Assume that

$$0 < |x - \bar{x}| < (2\gamma_2)^{-1/2}. \tag{2.49}$$

By (2.49), (2.44), (2.38) and (2.39)

$$\begin{aligned}
f_2(x) &= f_1(x) + \gamma_3(e, x - \bar{x}) = f_0(\bar{x}) + |x - \bar{x}|^2 + \gamma_3(e, x - \bar{x}), \\
g_2(x) &= g_1(x) + \gamma_3(e, \bar{x} - x) = g_0(\bar{x}) + |x - \bar{x}|^2 + \gamma_3(e, \bar{x} - x).
\end{aligned}$$

Together with (2.11), (2.43) and (2.23) these equalities imply that

$$\begin{aligned}
\max\{f_2(x), g_2(x)\} &= f_0(\bar{x}) + |x - \bar{x}|^2 + \max\{\gamma_3(e, x - \bar{x}),\ \gamma_3(e, \bar{x} - x)\} \\
&\geq f_0(\bar{x}) + |x - \bar{x}|^2 = \max\{f_2(\bar{x}), g_2(\bar{x})\} + |x - \bar{x}|^2 \\
&> \max\{f_2(\bar{x}), g_2(\bar{x})\}
\end{aligned}$$

and (2.47) is true.

Therefore (2.47) is true in all the cases. It follows from the choice of γ_3, (2.41), (2.43) and (2.46) that

$$(f_2, g_2) \in (\mathcal{U} \cap \mathcal{M}_0) \setminus G.$$

Proposition 2.1 is proved with $(\tilde{f}, \tilde{g}) = (f_2, g_2)$.

References

1. Aubin, J.P., Ekeland, I.: Applied Nonlinear Analysis. Wiley Interscience, New York (1984)
2. Demyanov, V.F., Malozemov, V.N.: Introduction to Minimax. Dover Publications Inc., New York (1990)
3. Demyanov, V.F., Rubinov, A.M.: Constructive Nonsmooth Analysis. Peter Lang, Frankfurt am Main (1995)
4. Demyanov, V.F., Vasil'ev, V.L.: Nondifferentiable Optimization. Translations Series in Mathematics and Engineering. Optimization Software Inc., New York (1985)
5. Nitecki, Z.: Differentiable Dynamics. An Introduction to the Orbit Structure of Diffeomorphisms. The MIT Press, Cambridge (1971)
6. Zaslavski, A.J.: Nondifferentiability of a cost function at a point of minimum in minimax problems. Optimization **54**, 507–516 (2005)
7. Zaslavski, A.J.: Genericity and minimax optimization. Math. Inequal. Appl. **8**, 717–726 (2005)
8. Zaslavski, A.J.: Turnpike Properties in the Calculus of Variations and Optimal Control. Springer, New York (2006)
9. Zaslavski, A.J.: On generic nondifferentiability of maximum type functions at a point of minimum. Pac. J. Optim. **3**, 213–224 (2007)